valid and higher powers of $(\xi q)^2$ must be included. (Unfortunately, the inclusion of these higher powers does not actually remove the divergence problem.) With this larger cutoff, one obtains

$$\delta\tau=2.1\tau_0 \ln[(T_{c0}+2\delta T_c)/2\delta T_c].$$

Fortunately, for our purposes, the two estimates are about the same if the shift of T_c is significant. They are exactly the same if $\delta\tau=\frac{1}{2}$, which requires $\tau_0=\frac{1}{3}$ or $R_\square=2.2\times10^4\ \Omega$. This value of the resistance is more than an order of magnitude larger than that where the $\delta\tau$ in our films is observed to be $\frac{1}{2}$. For our dirtiest films $R_\square\sim3\times10^3\ \Omega$, $\tau_0\sim0.05$, and the two estimates give a shift of T_c due to fluctuations $\delta T_c=\frac{1}{2}°$ and $\delta T_c=1°$.

V. CONCLUSION

We have measured the transition temperature for Pb films of varying thickness and found it to decrease substantially as the thickness decreases, vanishing for film thicknesses of the order of 10 Å. We have discussed various influences on T_c which may occur in very thin films and decided that the most important was interaction of the metal film with the substrate. Our understanding of the nature of these thin films is still rather qualitative. Further experiments, particularly tunneling experiments to extract the phonon spectrum and electron-phonon coupling strength, would add a great deal to our understanding.

ACKNOWLEDGMENTS

We have benefitted from many discussions with our colleagues. We are especially indebted to Professor A. Paskin, Professor R. D. Parks, and Dr. H. J. Lee. Besides many discussions, Professor Paskin contributed to the section on size quantization in small particles and Professor Parks collaborated on some of the initial experimental work. We are also indebted to G. Hrabak and T. Arns for expert technical assistance in the design and construction of the apparatus.

PHYSICAL REVIEW B VOLUME 1, NUMBER 3 1 FEBRUARY 1970

Superconducting Surface Sheath of a Semi-Infinite Half-Space and its Instability due to Fluctuations*

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(Received 15 September 1969)

The surface sheath of a semi-infinite half-space for applied magnetic fields $H_0\geq H_{c2}$ and Ginzburg-Landau κ values ≥0.707 was investigated with regard to its current-carrying capacity and stability. It is found that infinitesimally three-dimensional fluctuations of the order parameter and the vector potential make the surface sheath unstable and the total stable transport current zero.

I. INTRODUCTION

In recent years, Abrikosov,[1] Park,[2] Christiansen,[3] and Christiansen and Smith[4] have calculated for certain specific cases the maximum transport current of the surface sheath of a semi-infinite half-space. Their calculations are based on the Ginzburg-Landau (GL) equations, and the maximum current (or critical current) is defined as that current at which solutions of the GL equations cease to exist. Experiments seem to indicate that these theoretical values are orders of magnitude too large.[5–10]

A different approach was taken in Ref. 11. In order to obtain reasonable agreement with experiments, the critical current J_c was defined as that current which raises the total energy of the superconducting specimen to that of the normal state.[11,12] Various experimenters[5–8] seem to get fair agreement with this definition of the critical current. The critical current, however, is theoretically size-dependent. For example, J_c is proportional to $R^{-1/2}$, where R is the radius of a long cylinder, and the critical current is that which circulates on the surface around the axis of a cylinder when the applied field H_0 is parallel to the axis of the cylinder. The $R^{-1/2}$ dependence has been questioned by a number of

* Based on work supported by the Metallurgy Branch, Research Division, U. S. Atomic Energy Commission, under Contract No. AT(04-3)-701.

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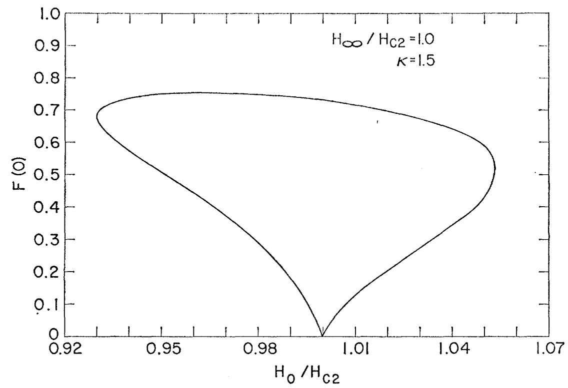

FIG. 1. Shown is the amplitude of the order parameter $F(0)$ at the surface of a semi-infinite half-space as a function of the applied field H_0 when the GL parameter $\kappa=1.5$ and $H_\infty=H_{c2}$, where H_{c2} is the bulk nucleation field ($\sqrt{2}\kappa H_c$) and H_∞ is the magnetic field inside the semi-infinite half-space far from the surface at which the surface sheath exists. $F(0)$ is a solution of Eqs. (4) and (5).

authors,[9,10,13] but no conclusive experimental evidence has yet been published which would indicate that J_c is not size-dependent. If the conclusion that $J_c \propto R^{-1/2}$ is correct, then $J_c=0$ for a semi-infinite half-space. As solutions of the GL equations for the semi-infinite half-space exist, which seem to indicate otherwise, the discrepancy between experiment and theory cannot be reconciled unless the GL solutions obtained for the semi-infinite half-space are unstable. It is the purpose of the present investigation to prove that infinitesimally three-dimensional fluctuations of the order parameter and the vector potential make the surface sheath of a semi-infinite half-space unstable, hence $J_c=0$.

II. GINZBURG-LANDAU AND FLUCTUATION EQUATIONS

In order to find if the superconducting surface sheath of a semi-infinite half-space is stable or unstable, we have to obtain first the detailed solutions of the GL equations for the superconducting surface sheath. The GL equations are the first variational equations of the total energy Ω of the superconductor with respect to the order parameter $\Psi(x,y,z)$ and the vector potential $\mathbf{A}(x,y,z)$. The second variation of Ω, which is $\delta^2\Omega$, leads to a number of equations which determine the fluctuations of Ψ and A, namely, $\delta\Psi$ and $\delta\mathbf{A}$. The functions $\delta\Psi$ and $\delta\mathbf{A}$ and their boundary conditions determine if $\delta^2\Omega \gtreqless 0$, and thus decide whether Ψ and $\mathbf{A}$ correspond to stable or unstable solutions.

The variational equations have been derived by Kramer[14] and somewhat generalized in Ref. 15. They are similar to the perturbation equations of Christiansen and Smith.[4] General solutions for the stability limit of the superheated Meissner state were obtained in Ref. 15, and we shall use here the same notation as introduced by Kramer[14] and used in Ref. 15.

The GL equations in the usual GL normalization are

$$\nabla^2 F=\kappa^2(F^2+\mathbf{Q}^2-1)F, \quad (1)$$

$$\text{curl curl}\mathbf{Q}=-F^2\mathbf{Q}, \quad (2)$$

where the order parameter

$$\Psi=F(x,y,z)e^{i\varphi(x,y,z)}; \quad \mathbf{Q}=\nabla\varphi/\kappa-\mathbf{A};$$

$\mathbf{A}$ is the vector potential; $\kappa=\lambda/\xi$; $\lambda(T)$ is the low-field penetration depth; $\xi(T)$ is the coherence length; x, y, and z are normalized by λ; H is normalized by $\sqrt{2}H_c$; and $\mathbf{H}=\text{curl}\mathbf{A}=-\text{curl}\mathbf{Q}$. At the boundary surface $H=H_0$ (H_0 is the external magnetic field) and $\partial F/\partial\mathbf{n}=0$, where $\mathbf{n}$ is normal to the surface. The variations of F and $\mathbf{Q}$, namely, δF and $\delta\mathbf{Q}$ are defined by the symbols $f(x,y,z)$ and $\mathbf{q}(x,y,z)$. The second variation of the Gibbs free energy Ω is

$$\delta^2\Omega=\int dV\left\{[3F^2+\mathbf{Q}^2-1]f^2+\left(\frac{\nabla f}{\kappa}\right)^2 +4Ff\mathbf{Q}\cdot\mathbf{q}+F^2\mathbf{q}^2+(\text{curl}\mathbf{q})^2\right\}, \quad (3)$$

where the integral of Eq. (3) is to be extended over all space. When $\delta^2\Omega>0$, the solution is stable and when $\delta^2\Omega<0$ it is unstable. Thus the stability limit is determined by $\delta^2\Omega=0$ (a quadratic functional has a minimum at zero). In order to minimize $\delta^2\Omega$ with respect to the functions f and $\mathbf{q}$ one finds the Euler-Lagrange equations of f and $\mathbf{q}$ from $\delta^2\Omega$ for a fixed set of the equilibrium functions F and $\mathbf{Q}$ (H_0 and κ are assumed to be constant). This is done in Refs. 14 and 15 when F and $\mathbf{Q}$ are specialized to the semi-infinite half-space.

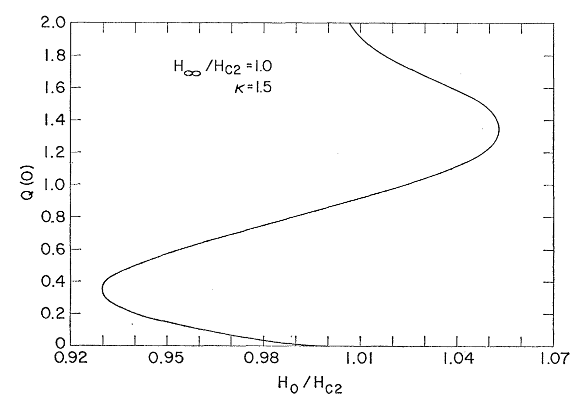

FIG. 2. Shown is the "superfluid velocity" $Q(0)$ at the surface of the semi-infinite half-space as a function of the applied field H_0 when the GL parameter $\kappa=1.5$ and $H_\infty=H_{c2}$, where H_{c2} is the bulk nucleation field ($\sqrt{2}\kappa H_c$) and H_∞ is the magnetic field inside the semi-infinite half-space far from the surface at which the surface sheath exists. $Q(0)$ is a solution of Eqs. (4) and (5). There are three solutions when $H_0/H_{c2}=1$: $F(0)=0$ and $Q(0)=0$; $F(0)=0$ and $Q(0)=\infty$; $F(0)=0.731$ and $Q(0)=0.863$.

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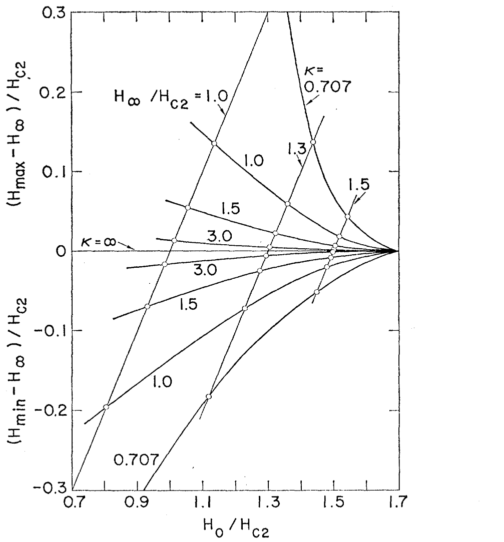

FIG. 3. Shown is the difference between H_{max} (H_{min}) and H_∞ as a function of the applied field H_0, where H_{max} (H_{min}) is defined as that field at which solutions to the GL equations cease to exist when H_0 is increased (decreased) beyond H_{max} (H_{min}). H_∞ is the magnetic field in the bulk of the metal. The points are calculated results for various κ and H_∞/H_{c2} values which are used as parameters in this plot.

We consider a semi-infinite half-space ($x\geq 0$) which is filled with a metal which is superconducting near $x=0$ and normal at $x=\infty$. The magnetic field **H** is parallel to the z direction and is defined by the superfluid velocity $\mathbf{Q}=(0;Q_y(x);0)$. Because of symmetry considerations we assume that $F=F(x)$. Then Eqs. (1) and (2) reduce to

$$\frac{d^2F}{dx^2}=\kappa^2F[F^2+Q^2-1], \tag{4}$$

$$\frac{d^2Q}{dx^2}=F^2Q, \tag{5}$$

where $Q=Q_y(x)$. Then Eq. (3) becomes

$$\delta^2\Omega=\int_0^\infty dx\{\tilde{f}A+\tilde{q}_yB+\tilde{q}_xC\}+\frac{1}{\kappa^2}\tilde{f}\frac{d\tilde{f}}{dx}\bigg|_0^\infty+\frac{F^2}{F^2+k_y^2}\tilde{q}_y\frac{d\tilde{q}_y}{dx}\bigg|_0^\infty, \tag{6}$$

where A, B, and C are the Euler-Lagrange equations:

$$A\equiv-\frac{1}{\kappa^2}\frac{d^2\tilde{f}}{dx^2}+\left[3F^2+Q^2+\left(\frac{k_y}{\kappa}\right)^2-1\right]\tilde{f}+2FQ\tilde{q}_y=0, \tag{7}$$

$$B\equiv-\frac{d}{dx}\left[\frac{F^2}{F^2+k_y^2}\frac{d\tilde{q}_y}{dx}\right]+F^2\tilde{q}_y+2FQ\tilde{f}=0, \tag{8}$$

$$C\equiv-k_y\frac{d\tilde{q}_y}{dx}+(F^2+k_y^2)\tilde{q}_x=0. \tag{9}$$

$\tilde{f}$, $\tilde{q}_y$, and $\tilde{q}_x$ are the amplitudes of the fluctuations of F and **Q**, and k_y is the wave vector in the y direction of a particular mode. It is shown in detail in Ref. 15 that the three-dimensional fluctuation equations for the semi-infinite half-space reduce to the above simple equations.

Before we can find solution for the critical fluctuation amplitudes $\tilde{f}$ and $\tilde{q}_y$, we have to know the exact solutions of Eqs. (4) and (5). These equations were solved with the following boundary conditions: $(dF/dx)_0=0$ ($x<0$ is considered vacuum); $F(\infty)=0$; $(dQ/dx)_0=-H_0$, and $(dQ/dx)_\infty=-H_\infty$, where H_∞ is the value of the internal field at $x=+\infty$. When Eq. (4) is integrated once with respect to F and Eq. (5) once with respect to Q between the limits $x=0$ and $x=\infty$ and the results are combined, one obtains

$$H_\infty^2-H_0^2=F^2(0)[1-Q^2(0)-\tfrac{1}{2}F^2(0)]. \tag{10}$$

Equation (10) was used to check the consistency of our results. Note also that the difference $H_\infty-H_0$ is proportional to the total surface current per unit length along the z direction and that this value depends only on $F(0)$, $Q(0)$, and (H_0+H_∞).

III. RESULTS OF GINZBURG-LANDAU EQUATIONS

As an example we show in Figs. 1 and 2 the solutions $F(0)$ and $Q(0)$ of Eqs. (3) and (4) for $\kappa=1.5$ when the magnetic field is equal to the bulk nucleation field H_{c2} ($=\sqrt{2}\kappa H_c$) and the external field H_0 was varied. Both $F(0)$ and $Q(0)$ are double values for a fixed value of H_0/H_{c2} and one expects intuitively that the lower branch of $F(0)$ in Fig. 1 is unstable when compared to the upper branch. H_0 can be increased only up to a field H_{max} above which no solution with the above boundary conditions exists for a fixed value of H_∞ and κ. Similarly, there is a minimum field H_{min} below which no solution exists. A similar conclusion can be drawn from Fig. 2. The upper branch of $F(0)$ corresponds to the $Q(0)$ values with positive slope in Fig. 2. Similar

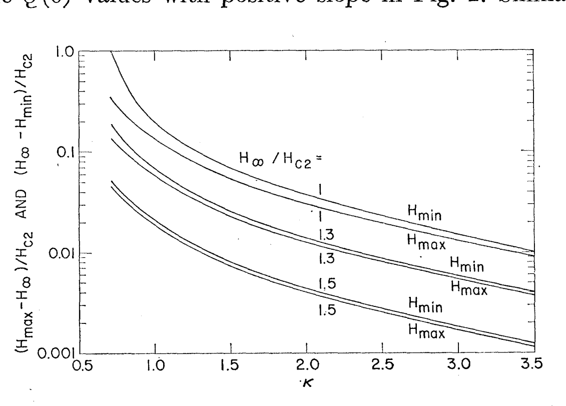

FIG. 4. This plot is similar to Fig. 3 except that the logarithm of $(H_{max}-H_\infty)$ and $(H_\infty-H_{min})$ is plotted as a function of κ with H_∞/H_{c2} as a parameter. $(H_\infty-H_{min})$ is always larger than $(H_{max}-H_\infty)$ for a fixed value of κ and H_∞/H_{c2}.

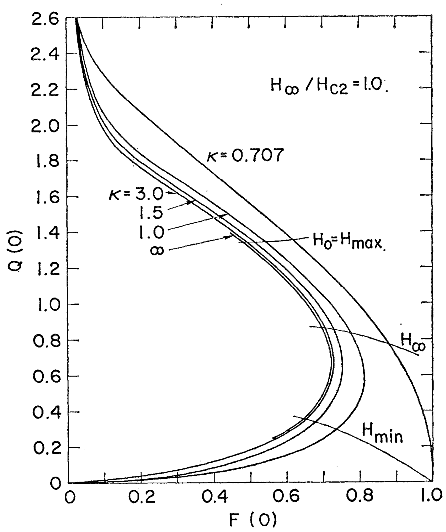

FIG. 5. $F(0)$ and $Q(0)$, which are solutions of Eqs. (4) and (5) at the surface of a semi-infinite half-space, are plotted for various values of κ when $H_\infty/H_{c2}=1.0$. The curve for $\kappa=\infty$ was obtained from extrapolations (see text). The values of $F(0)$ and $Q(0)$ at $H_0=H_{\max}$, at $H_0=H_\infty$ and at $H_0=H_{\min}$ are indicated by thin lines. When $F(0)$, $Q(0)$ and H_∞ are known for a given κ value, H_0 can be calculated from Eq. (10).

results were obtained for $\kappa=0.707$, 1, 1.5, 3 and values of $H_\infty/H_{c2}=1.0$, 1.3, and 1.5. These 12 cases were investigated in detail. When $H_0=H_\infty$ there are two values of $F(0)$, one of which is always zero. To $F(0)=0$ correspond two $Q(0)$ values, namely, $Q(0)=0$ and $Q(0)=\infty$. The maximum field $H_{\max}$ and the minimum field $H_{\min}$ are summarized in Figs. 3 and 4 for various H_∞/H_{c2} and κ values. In Fig. 3 the circles correspond to the calculated values and the logarithmic plot of Fig. 4 emphasizes the larger κ values and the higher fields. The asymmetry of $F(0)$ which can be seen in Fig. 1 and also that of $Q(0)$ (Fig. 2) reflects itself clearly into Figs. 3 and 4.

Figures 5–7 summarize the results of $F(0)$ and $Q(0)$ for various κ and H_∞/H_{c2} values. The values of $F(0)$ and $Q(0)$ at $H_0=H_{\max}$, $H_0=H_\infty$, and $H_0=H_{\min}$ are indicated. The curves for $\kappa=\infty$ were obtained by plotting $F(0)$ and $Q(0)$ at $H_{\max}$ and $H_{\min}$, as a function of $H_{\max}/H_{c2}$ and $H_{\min}/H_{c2}$ for all available κ values for $H_\infty/H_{c2}=$ const and extrapolating these curves to $H_{\max}=H_\infty$ and $H_{\min}=H_\infty$. $F(0)$ and $Q(0)$ for $\kappa=\infty$ and $H_0=H_\infty$ were obtained by plotting $F(0)$ and $Q(0)$ as a function of $1/\kappa$ at $H_0=H_\infty$ for all available κ values for $H_\infty/H_{c2}=$ const and extrapolating these curves to $1/\kappa\rightarrow 0$. When $H_0\rightarrow H_{c3}(=1.695H_{c2})$ the value of $F(0)\rightarrow 0$ and $Q(0)\rightarrow 1$. This can be seen readily from Figs. 5–7 and this is consistent with Eq. (10).

Assuming that the above solutions for the semi-infinite half-space were stable, one would interpret the maximum or critical current as that which is determined by $H_{\max}$ or $H_{\min}$, respectively. The critical current per unit length along the z direction is then, in Gaussian units

$$J_c=\int_0^\infty j(x)dx=\frac{c}{4\pi}|H_{\max,\min}-H_\infty|\,. \qquad (11)$$

As can be seen from Figs. 3 and 4, these currents are quite appreciable. They are orders of magnitude larger than one observes on small specimens.[5–10] In Sec. IV we shall investigate the stability of the solutions $F(x)$ and $Q(x)$.

IV. RESULTS OF FLUCTUATION EQUATIONS

Equation (6) is the second variation of the total free energy and Eqs. (7)–(9) are the equations which describe the critical fluctuations of F and Q which determine the stability limit $\delta^2\Omega=0$, provided one can find solutions of $\tilde{f}$ and $\tilde{q}_y$ with the desired boundary conditions which make the constants of integration of Eq. (6) equal to zero. When $\delta^2\Omega>0$ the solutions of $F(x)$ and $Q(x)$ are stable and when $\delta^2\Omega<0$ they are unstable. It is thus the aim to find solutions of Eqs. (7) and (8) which make the constants of integration of Eq. (6) equal to zero.

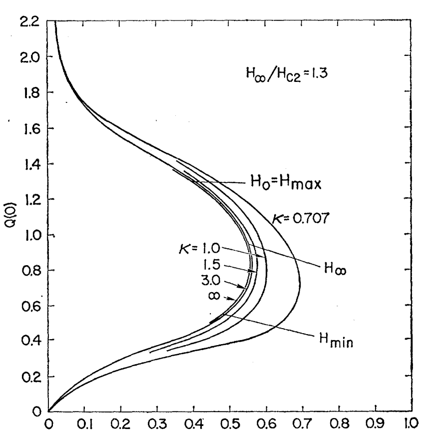

FIG. 6. This figure is similar to Fig. 5 except that $H_\infty/H_{c2}=1.3$.

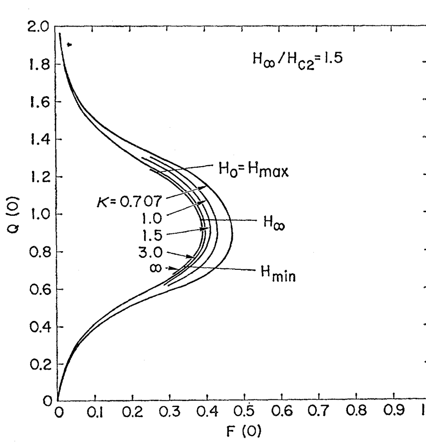

FIG. 7. This figure is similar to Fig. 5 except that $H_\infty/H_{c2}=1.5$.

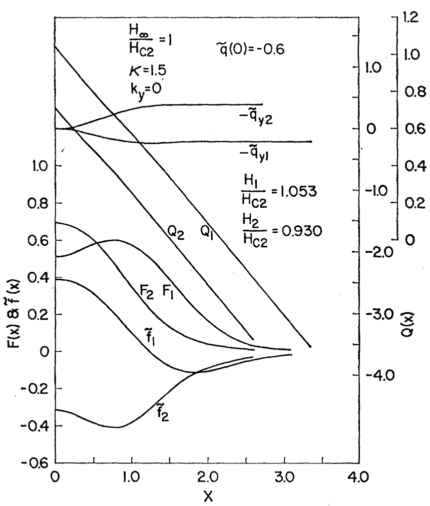

FIG. 8. Shown are the solutions of Eqs. (4) and (5) and of the critical fluctuation Eqs. (7) and (8) for $\kappa=1.5$ and $H_0=H_{\max}\equiv H_1$ and $H_0=H_{\min}\equiv H_2$ when the wave vector of the fluctuation $k_y=0$ (infinite wavelength). For these particular fields ($H_{\max}$ and $H_{\min}$) $k_y=0$ is a solution which makes the second variation of the energy $\delta^2\Omega=0$ [Eq. (6)]. x is in GL normalized units.

Let us first consider the case when $k_y=0$. Possible solutions for $\tilde{f}$ and $\tilde{q}_y$ which satisfy Eqs. (7) and (8) are: $\tilde{f}=dF/dx$ and $\tilde{q}_y=dQ/dx$; $\tilde{f}=\partial F(x,F(0))/\partial F(0)$ and $\tilde{q}_y=\partial Q(x,F(0))/\partial F(0)$; $\tilde{f}=\partial F(x,Q(0))/\partial Q(0)$ and $\tilde{q}_y=\partial Q(x,Q(0))/\partial Q(0)$; $\tilde{f}=\partial F(x,H_0)/\partial H_0$ and $\tilde{q}_y=\partial Q(x,H_0)/\partial H_0$. The derivative with respect to x does not make $\delta^2\Omega=0$. Thus we consider the other three possibilities, all of which give the same final answer. For example:

$$\delta^2\Omega=\frac{1}{\kappa^2}\frac{\partial F}{\partial F(0)}\frac{\partial^2 F}{\partial F(0)\partial x}\bigg|_0^\infty+\frac{\partial Q}{\partial F(0)}\frac{\partial^2 Q}{\partial F(0)dx}\bigg|_0^\infty. \quad (12)$$

As $\partial F/\partial x=0$ at $x=0$ and $x=\infty$, Eq. (12) becomes

$$\delta^2\Omega=-\frac{\partial Q}{\partial F(0)}\frac{\partial H}{\partial F(0)}\bigg|_0^\infty=-\frac{\partial Q}{\partial F(0)}\bigg|_\infty\frac{\partial H}{\partial F(0)}\bigg|_{H=H_\infty}+\frac{\partial Q}{\partial F(0)}\bigg|_0\frac{\partial H}{\partial F(0)}\bigg|_{H=H_0}. \quad (13)$$

When H_∞ is kept constant as in Figs. 1, 2, and 5–7, then it can be seen readily from these figures that $\delta^2\Omega=0$ for all κ values when $H_0=H_{\max}$ and $H_0=H_{\min}$.

Detailed solutions of Eqs. (4), (5), (7), and (8) are shown in Fig. 8 for $H_\infty/H_{c2}=1$, $\kappa=1.5$, and $k_y=0$ when $H_0=H_{\max}(\equiv H_1)$ and $H_0=H_{\min}(\equiv H_2)$. These were obtained on an analog computer by a similar method as described in detail in Ref. 15. As Eqs. (7) and (8) are coupled and linear in $\tilde{f}$ and $\tilde{q}_y$, the ratio $\tilde{f}(0)/\tilde{q}_y(0)$ is not arbitrary and has to be adjusted by trial and error until the desired boundary conditions are satisfied, provided they can be satisfied at all. Only then are the solutions for $\tilde{f}$ and $\tilde{q}_y$ those which make $\delta^2\Omega=0$. In Fig. 8 $(d\tilde{f}/dx)_0=0$ and $\tilde{f}(\infty)=0$, and also $(d\tilde{q}_y/dx)=0$ at $x=0$ and $x=\infty$. It therefore follows from Eq. (6) that $\delta^2\Omega=0$ and hence F and Q are unstable for at least one value of k_y at $H_{\min}$ and $H_{\max}$.

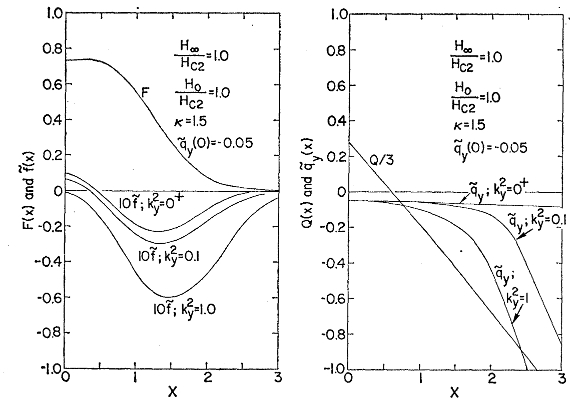

FIG. 9. Shown are the solutions of Eqs. (4) and (5) and of the critical fluctuation Eqs. (7) and (8) when $\kappa=1.5$ and $H_0/H_{c2}=H_\infty/H_{c2}=1$ for various values of the wave vector k_y. The value $k_y=0^+$ means $0<k_y^2\ll 0.1$. When $k_y^2\geq 0$ the term $F^2\tilde{q}_y(d\tilde{q}_y/dx)/(F^2+k_y^2)$ in Eq. (6) for $x=\infty$ is zero and thus $\delta^2\Omega=0$. x is in GL normalized units.

When $k_y^2>0$, solutions with $(d\tilde{q}_y/dx)_\infty=0$ could not be found. Although the desired boundary conditions of $\tilde{f}(x)$ could always be satisfied for the proper ratio of $\tilde{f}(0)/\tilde{q}_y(0)$, the value of $|d\tilde{q}_y/dx|_\infty$ is always larger than zero except when $k_y=0$.

Figure 9 shows the results of F, Q, $\tilde{f}$, and $\tilde{q}_y$ for $H_\infty/H_{c2}=1.0$, $H_0/H_{c2}=1.0$, and $\kappa=1.5$. Figure 10 shows similar results for $H_\infty/H_{c2}=1.3$, $H_0/H_{c2}=1.3$ and $\kappa=1.5$. It can be seen that neither $\tilde{q}_y$ nor $d\tilde{q}_y/dx$ are zero when $x\rightarrow\infty$ and $k_y>0$.

One can find asymptotic expressions of F, Q, $\tilde{f}$, and $\tilde{q}_y$ for $x\gg 1$. With $Q=-H_\infty x$ it follows from Eq. (1) that $F=F_0e^{-\alpha x^2}$, where $\alpha=\frac{1}{2}\kappa H_\infty$. Equations (7) and (8) can then be satisfied by the following expressions:

$$\tilde{f}=f_0(ax+e^{k_yx})e^{-\alpha x^2}, \quad (14)$$

$$\tilde{q}=q_\infty e^{k_yx}+q_0e^{-2\alpha x^2}, \quad (15)$$

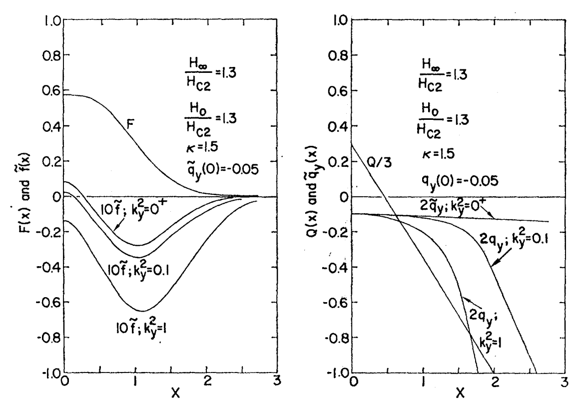

FIG. 10. This figure is similar to Fig. 9 except that $H_0/H_{c2}=H_\infty/H_{c2}=1.3$.

where F_0, f_0, a, q_∞, and q_0 are parameters which are interrelated. The asymptotic behavior of Eqs. (14) and (15) is consistent with Figs. 8–10. Similar expressions were first suggested by Kramer.[16] When Eqs. (14) and (15) are substituted into $F^2\tilde{q}_y(d\tilde{q}_y/dx)/(F^2+k_y^2)$ one finds that this term approaches zero for $x\rightarrow\infty$ for all values of $k_y\geq 0$. Hence it follows from Eq. (6) that $\delta^2\Omega=0$ for all values of $k_y\geq 0$.

Thus the surface sheath of a semi-infinite half-space is unstable as it is sufficient for one mode to induce critical fluctuations because the modes are not coupled. We have found similar results for all the κ values investigated and all fields between $H_{\min}$ and $H_{\max}$. The $\tilde{f}$ and $\tilde{q}_y$ function are very similar in nature as those shown in Figs. 8–10. Hence we conclude that the superconducting surface sheath of a semi-infinite half-space is unstable for all magnetic fields and all κ values and therefore the total stable current in the sheath [Eq. (11)] is zero for a semi-infinite half-space.

The general behavior of the $\tilde{f}(x)$ and the $\tilde{q}_y(x)$ functions of the surface sheath are different from those of the Meissner state.[15] For the Meissner state $\tilde{q}_y(\infty)$ is always zero and $\tilde{f}(x)$ and $\tilde{q}_y(x)$ are always of opposite sign. For the surface sheaths $q_y(\infty)$ is not zero, in general, and $\tilde{f}(x)$ may reverse sign so that $\tilde{f}(x)/\tilde{q}_y(x)$ may have either sign. $\tilde{f}(\infty)$ is always zero for both cases.

V. CONCLUSIONS

The general solutions of the surface sheath of a semi-infinite half-space for applied magnetic fields $H_0\geq H_{c2}$ and $\kappa>0.707$ were found and investigated with regard to their current-carrying capacity and their stability. It is found that infinitesimally three-dimensional fluctuations of the order parameter and the vector potential make the surface sheath of a semi-infinite half-space unstable. We have shown that at least one critical fluctuation exists for all κ values and all magnetic fields for which the second variation of the total energy $\delta^2\Omega$ [Eq. (6)] ceases to be positive definite. Thus the surface sheath of a semi-infinite half-space cannot carry a transport current as the sheath itself is unstable, even when the total transport current is zero. This conclusion is consistent with that of the critical current J_c as defined in Ref. 11, where it was shown from energy considerations that J_c must be size dependent and approach zero for an infinite specimen. When the specimen is finite, one would expect that the sheath is stable and can carry a finite transport current, but no rigid proof from first principle exists at present. The detailed solutions of $F(0)$, $Q(0)$, $H_{\max}$ and $H_{\min}$ as summarized in Figs. 1–7 should be useful in obtaining an approximate solution of the same problem for a finite specimen.

Up to the present it has always been assumed in the literature that the one-dimensional sheath on a semi-infinite half-space is a stable configuration. This calculation shows that this assumption is not correct if the sheath has the above simple equilibrium configuration. However, the above proof does not rule out the possibility of a sheath of a different configuration.[17]

Note added in proof. It follows from Eq. (15) that when $k_y>0$, the value of $\tilde{q}_y$ does not remain a small perturbation with respect to Q for large values of x. On first sight this seems to contradict the perturbation method used in the variational approach. Instead of investigating the variations of F and Q, one can investigate the variations of F and the current density $j(x)$. One finds that $j(x)$ and the fluctuations of $j(x)$ for large values of x converge and are well behaved. Thus there exists no violation of the perturbation method.

ACKNOWLEDGMENTS

The authors thank Dr. L. Kramer for reading the manuscript and for helpful suggestions, and Dr. J. G. Park for his comments.

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